3.3 Vector Addition and Subtraction: Analytical Methods

Analytical methods of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

Resolving a Vector into Perpendicular Components

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like $\mathbf{A}$ in Figure 3.26, we may wish to find which two perpendicular vectors, $\mathbf{A}_x$ and $\mathbf{A}_y$, add to produce it.

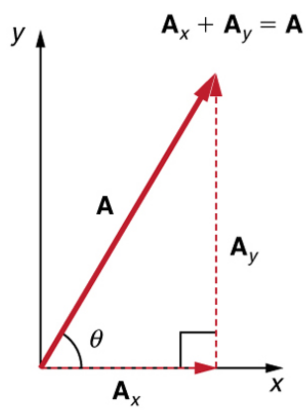

Figure 3.26 The vector $\mathbf{A}$, with its tail at the origin of an *x*, *y*-coordinate system, is shown together with its *x*- and *y*-components, $\mathbf{A}_x$ and $\mathbf{A}_y$. These vectors form a right triangle. The analytical relationships among these vectors are summarized below.

$\mathbf{A}_x$ and $\mathbf{A}_y$ are defined to be the components of $\mathbf{A}$ along the *x*- and *y*-axes. The three vectors $\mathbf{A}$, $\mathbf{A}_x$, and $\mathbf{A}_y$ form a right triangle:

$$\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}. \tag{3.3}$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if $\mathbf{A}_x = 3 \text{ m}$ east, $\mathbf{A}_y = 4 \text{ m}$ north, and $\mathbf{A} = 5 \text{ m}$ north-east, then it is true that the vectors $\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}$. However, it is *not* true that the sum of the magnitudes of the vectors is also equal. That is,

$$3 \text{ m} + 4 \text{ m} \neq 5 \text{ m} \tag{3.4}$$

Thus,

$$A_x + A_y \neq A \tag{3.5}$$

If the vector $\mathbf{A}$ is known, then its magnitude A (its length) and its angle θ (its direction) are known. To find A_x and A_y, its *x*- and *y*-components, we use the following relationships for a right triangle.

$$A_x = A \cos\theta \tag{3.6}$$

and

$$A_y = A \sin\theta. \tag{3.7}$$

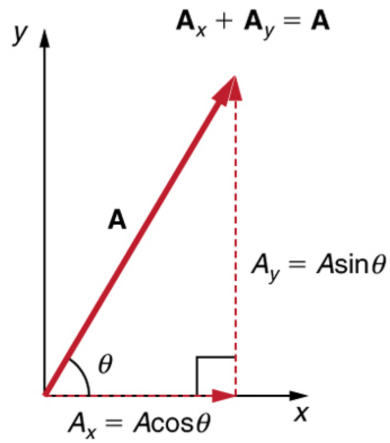

Figure 3.27 The magnitudes of the vector components $\mathbf{A}_x$ and $\mathbf{A}_y$ can be related to the resultant vector $\mathbf{A}$ and the angle θ with trigonometric identities. Here we see that $A_x = A\cos\theta$ and $A_y = A\sin\theta$.

Suppose, for example, that $\mathbf{A}$ is the vector representing the total displacement of the person walking in a city considered in Kinematics in Two Dimensions: An Introduction and Vector Addition and Subtraction: Graphical Methods.

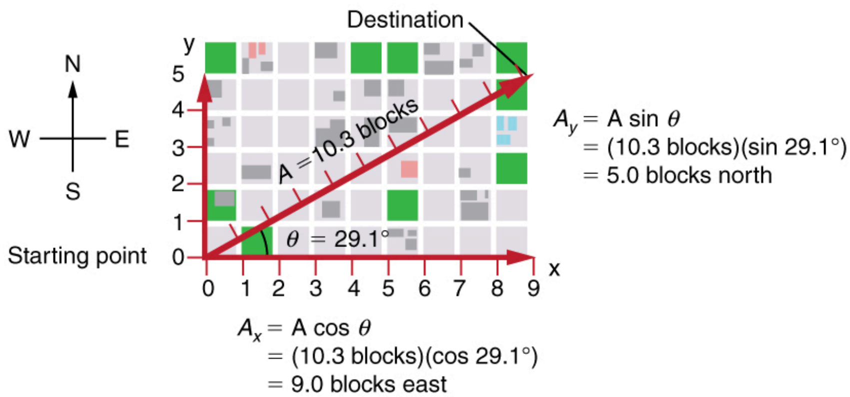

Figure 3.28 We can use the relationships $A_x = A\cos\theta$ and $A_y = A\sin\theta$ to determine the magnitude of the horizontal and vertical component vectors in this example.

Then $A = 10.3$ blocks and $\theta = 29.1^\circ$, so that

$$A_x = A\cos\theta = (10.3\text{ blocks})(\cos 29.1^\circ) = 9.0\text{ blocks} \tag{3.8}$$

$$A_y = A\sin\theta = (10.3\text{ blocks})(\sin 29.1^\circ) = 5.0\text{ blocks.} \tag{3.9}$$

Calculating a Resultant Vector

If the perpendicular components $\mathbf{A}_x$ and $\mathbf{A}_y$ of a vector $\mathbf{A}$ are known, then $\mathbf{A}$ can also be found analytically. To find the magnitude A and direction θ of a vector from its perpendicular components $\mathbf{A}_x$ and $\mathbf{A}_y$, we use the following relationships:

$$A = \sqrt{A_x{}^2 + A_y{}^2} \tag{3.10}$$

$$\theta = \tan^{-1}(A_y/A_x). \tag{3.11}$$

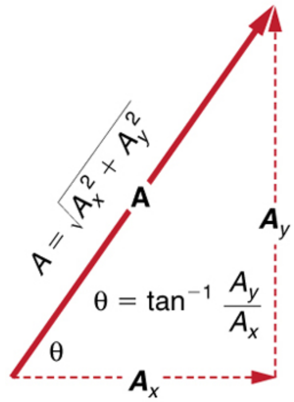

Figure 3.29 The magnitude and direction of the resultant vector can be determined once the horizontal and vertical components A_x and A_y have been determined.

Note that the equation $A = \sqrt{A_x^2 + A_y^2}$ is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For example, if A_x and A_y are 9 and 5 blocks, respectively, then $A = \sqrt{9^2+5^2}=10.3$ blocks, again consistent with the example of the person walking in a city. Finally, the direction is $\theta = \tan^{-1}(5/9)=29.1°$, as before.

Determining Vectors and Vector Components with Analytical Methods

Equations $A_x = A \cos \theta$ and $A_y = A \sin \theta$ are used to find the perpendicular components of a vector—that is, to go from A and θ to A_x and A_y. Equations $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = \tan^{-1}(A_y/A_x)$ are used to find a vector from its perpendicular components—that is, to go from A_x and A_y to A and θ. Both processes are crucial to analytical methods of vector addition and subtraction.

Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider Figure 3.30, in which the vectors $\mathbf{A}$ and $\mathbf{B}$ are added to produce the resultant $\mathbf{R}$.

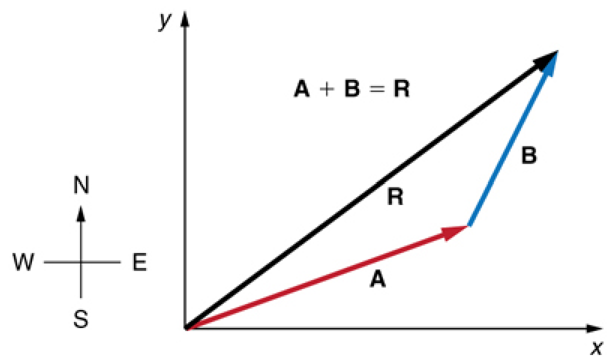

Figure 3.30 Vectors $\mathbf{A}$ and $\mathbf{B}$ are two legs of a walk, and $\mathbf{R}$ is the resultant or total displacement. You can use analytical methods to determine the magnitude and direction of $\mathbf{R}$.

If $\mathbf{A}$ and $\mathbf{B}$ represent two legs of a walk (two displacements), then $\mathbf{R}$ is the total displacement. The person taking the walk ends up at the tip of $\mathbf{R}$. There are many ways to arrive at the same point. In particular, the person could have walked first in the *x*-direction and then in the *y*-direction. Those paths are the *x*- and *y*-components of the resultant, $\mathbf{R}_x$ and $\mathbf{R}_y$. If we know $\mathbf{R}_x$ and $\mathbf{R}_y$, we can find R and θ using the equations $A = \sqrt{{A_x}^2 + {A_y}^2}$ and $\theta = \tan^{-1}(A_y/A_x)$. When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

Step 1. *Identify the x- and y-axes that will be used in the problem. Then, find the components of each vector to be added along* *the chosen perpendicular axes.* Use the equations $A_x = A \cos \theta$ and $A_y = A \sin \theta$ to find the components. In Figure 3.31, these components are A_x, A_y, B_x, and B_y. The angles that vectors $\mathbf{A}$ and $\mathbf{B}$ make with the x-axis are θ_A and θ_B, respectively.

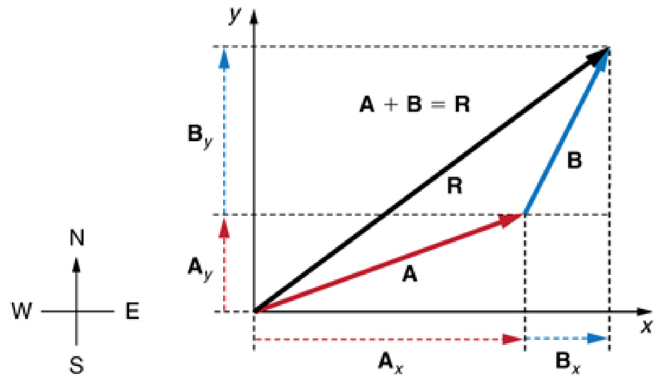

Figure 3.31 To add vectors $\mathbf{A}$ and $\mathbf{B}$, first determine the horizontal and vertical components of each vector. These are the dotted vectors $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{B}_x$ and $\mathbf{B}_y$ shown in the image.

Step 2. *Find the components of the resultant along each axis by adding the components of the individual vectors along that axis.* That is, as shown in Figure 3.32,

$$R_x = A_x + B_x \tag{3.12}$$

and

$$R_y = A_y + B_y. \tag{3.13}$$

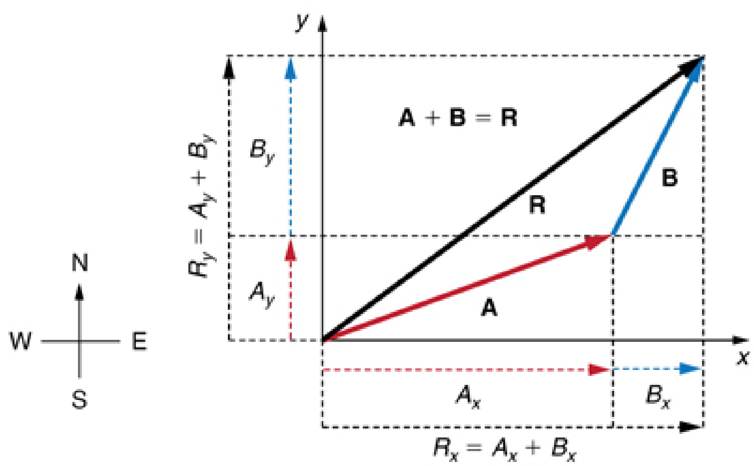

Figure 3.32 The magnitude of the vectors $\mathbf{A}_x$ and $\mathbf{B}_x$ add to give the magnitude R_x of the resultant vector in the horizontal direction. Similarly, the magnitudes of the vectors $\mathbf{A}_y$ and $\mathbf{B}_y$ add to give the magnitude R_y of the resultant vector in the vertical direction.

Components along the same axis, say the x-axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the y-axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of $\mathbf{R}$ are known, its magnitude and direction can be found.

Step 3. *To get the magnitude R of the resultant, use the Pythagorean theorem:*

$$R = \sqrt{R_x^2 + R_y^2}. \tag{3.14}$$

Step 4. *To get the direction of the resultant:*

$$\theta = \tan^{-1}(R_y/R_x). \tag{3.15}$$

The following example illustrates this technique for adding vectors using perpendicular components.

EXAMPLE 3.3

Adding Vectors Using Analytical Methods

Add the vector $\mathbf{A}$ to the vector $\mathbf{B}$ shown in Figure 3.33, using perpendicular components along the x- and y-axes. The x- and y-axes are along the east–west and north–south directions, respectively. Vector $\mathbf{A}$ represents the first leg of a walk in which a person walks 53.0 m in a direction 20.0° north of east. Vector $\mathbf{B}$ represents the second leg, a displacement of 34.0 m in a direction 63.0° north of east.

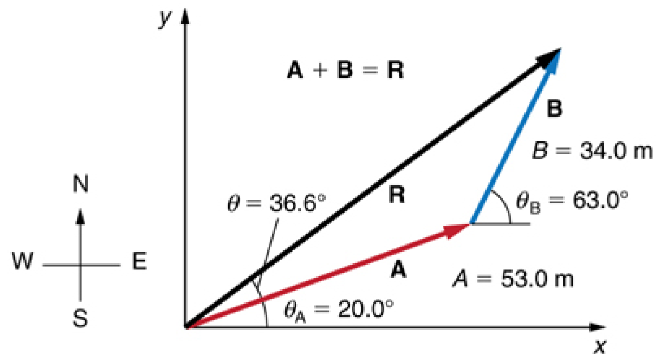

Figure 3.33 Vector $\mathbf{A}$ has magnitude 53.0 m and direction 20.0° north of the x-axis. Vector $\mathbf{B}$ has magnitude 34.0 m and direction 63.0° north of the x-axis. You can use analytical methods to determine the magnitude and direction of $\mathbf{R}$.

Strategy

The components of $\mathbf{A}$ and $\mathbf{B}$ along the x- and y-axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

Solution

Following the method outlined above, we first find the components of $\mathbf{A}$ and $\mathbf{B}$ along the x- and y-axes. Note that $A = 53.0 \text{ m}$, $\theta_{\text{A}} = 20.0^\circ$, $B = 34.0 \text{ m}$, and $\theta_{\text{B}} = 63.0^\circ$. We find the x-components by using $A_x = A \cos \theta$, which gives

$$\begin{aligned} A_x &= A \cos \theta_{\text{A}} = (53.0 \text{ m})(\cos 20.0^\circ) \\ &= (53.0 \text{ m})(0.940) = 49.8 \text{ m} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} B_x &= B \cos \theta_{\text{B}} = (34.0 \text{ m})(\cos 63.0^\circ) \\ &= (34.0 \text{ m})(0.454) = 15.4 \text{ m.} \end{aligned} \tag{3.17}$$

Similarly, the y-components are found using $A_y = A \sin \theta_{\text{A}}$:

$$\begin{aligned} A_y &= A \sin \theta_{\text{A}} = (53.0 \text{ m})(\sin 20.0^\circ) \\ &= (53.0 \text{ m})(0.342) = 18.1 \text{ m} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} B_y &= B \sin \theta_{\text{B}} = (34.0 \text{ m})(\sin 63.0^\circ) \\ &= (34.0 \text{ m})(0.891) = 30.3 \text{ m.} \end{aligned} \tag{3.19}$$

The x- and y-components of the resultant are thus

$$R_x = A_x + B_x = 49.8 \text{ m} + 15.4 \text{ m} = 65.2 \text{ m} \tag{3.20}$$

and

$$R_y = A_y + B_y = 18.1 \text{ m} + 30.3 \text{ m} = 48.4 \text{ m.} \tag{3.21}$$

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(65.2)^2 + (48.4)^2 \text{ m}} \quad (3.22)$$

so that

$$R = 81.2 \text{ m}. \quad (3.23)$$

Finally, we find the direction of the resultant:

$$\theta = \tan^{-1}(R_y/R_x) = +\tan^{-1}(48.4/65.2). \quad (3.24)$$

Thus,

$$\theta = \tan^{-1}(0.742) = 36.6^\circ. \quad (3.25)$$

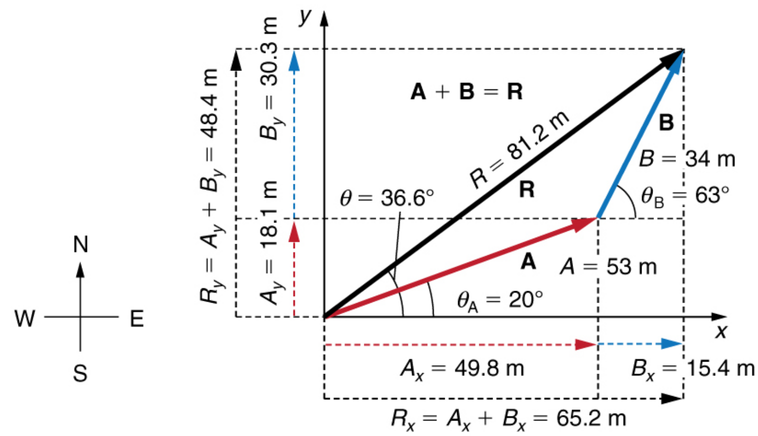

Figure 3.34 Using analytical methods, we see that the magnitude of $\mathbf{R}$ is 81.2 m and its direction is 36.6° north of east.

Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector. That is, $\mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (-\mathbf{B})$. Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition*. The components of $-\mathbf{B}$ are the negatives of the components of $\mathbf{B}$. The x- and y-components of the resultant $\mathbf{A} - \mathbf{B} = \mathbf{R}$ are thus

$$R_x = A_x + (-B_x) \quad (3.26)$$

and

$$R_y = A_y + (-B_y) \quad (3.27)$$

and the rest of the method outlined above is identical to that for addition. (See Figure 3.35.)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, Projectile Motion, is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.

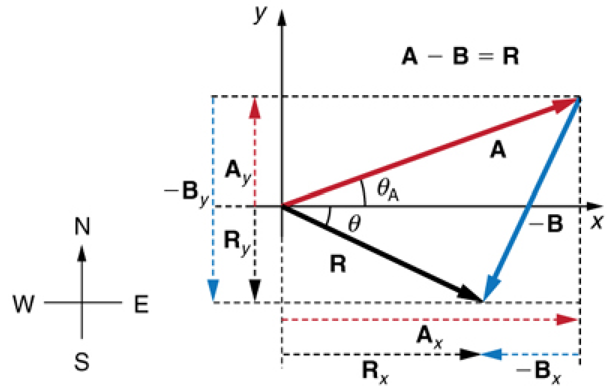

Figure 3.35 The subtraction of the two vectors shown in Figure 3.30. The components of $-\mathbf{B}$ are the negatives of the components of $\mathbf{B}$. The method of subtraction is the same as that for addition.

PHET EXPLORATIONS

Vector Addition

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats.

Click to view content (https://phet.colorado.edu/sims/vector-addition/vector-addition_en.html)

3.4 Projectile Motion

Projectile motion is the **motion** of an object thrown or projected into the air, subject to only the acceleration of gravity. The object is called a **projectile**, and its path is called its **trajectory**. The motion of falling objects, as covered in Problem-Solving Basics for One-Dimensional Kinematics, is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In this section, we consider two-dimensional projectile motion, such as that of a football or other object for which **air resistance** *is negligible*.

The most important fact to remember here is that *motions along perpendicular axes are independent* and thus can be analyzed separately. This fact was discussed in Kinematics in Two Dimensions: An Introduction, where vertical and horizontal motions were seen to be independent. The key to analyzing two-dimensional projectile motion is to break it into two motions, one along the horizontal axis and the other along the vertical. (This choice of axes is the most sensible, because acceleration due to gravity is vertical—thus, there will be no acceleration along the horizontal axis when air resistance is negligible.) As is customary, we call the horizontal axis the *x*-axis and the vertical axis the *y*-axis. Figure 3.36 illustrates the notation for displacement, where $\mathbf{s}$ is defined to be the total displacement and $\mathbf{x}$ and $\mathbf{y}$ are its components along the horizontal and vertical axes, respectively. The magnitudes of these vectors are *s*, *x*, and *y*. (Note that in the last section we used the notation $\mathbf{A}$ to represent a vector with components $\mathbf{A}_x$ and $\mathbf{A}_y$. If we continued this format, we would call displacement $\mathbf{s}$ with components $\mathbf{s}_x$ and $\mathbf{s}_y$. However, to simplify the notation, we will simply represent the component vectors as $\mathbf{x}$ and $\mathbf{y}$.)

Of course, to describe motion we must deal with velocity and acceleration, as well as with displacement. We must find their components along the *x*- and *y*-axes, too. We will assume all forces except gravity (such as air resistance and friction, for example) are negligible. The components of acceleration are then very simple: $a_y = -g = -9.80\ \text{m/s}^2$. (Note that this definition assumes that the upwards direction is defined as the positive direction. If you arrange the coordinate system instead such that the downwards direction is positive, then acceleration due to gravity takes a positive value.) Because gravity is vertical, $a_x = 0$. Both accelerations are constant, so the kinematic equations can be used.

Review of Kinematic Equations (constant a)

$$x = x_0 + \bar{v}t \tag{3.28}$$